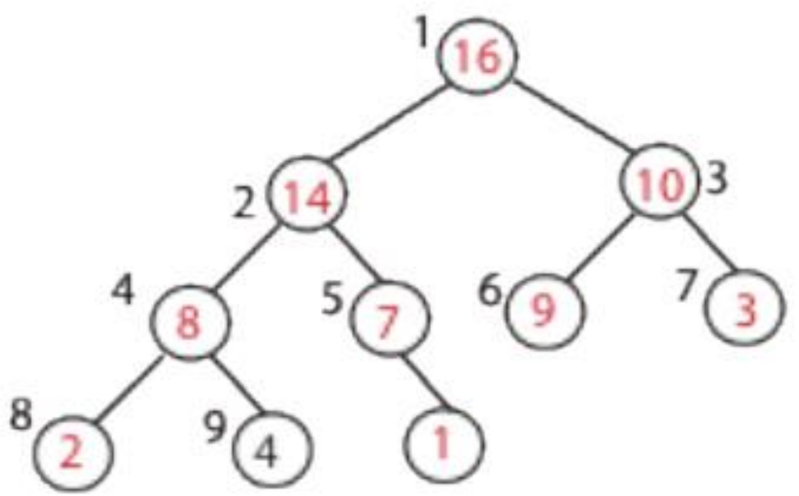
Algorithms

Lecture#4

Review of last lecture

- Heap : is a nearly complete binary tree. Of height ᶱ(lgn)

Max-Heap Property: The key of a node is $\geq$ than the keys of its children.



Review of last lecture…

Visualizing an Array as a Tree

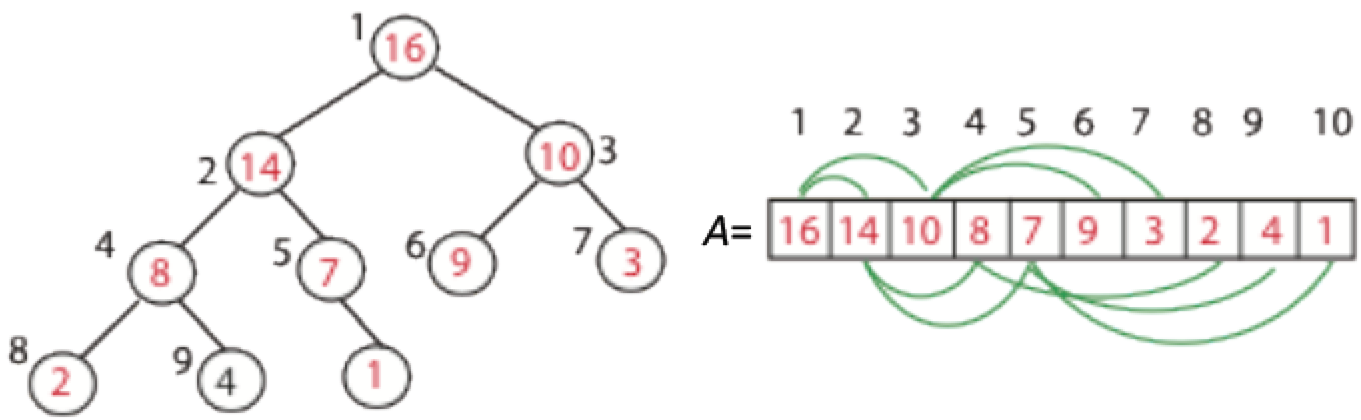
Root of tree: first element in the array, corresponding to index = 1

If a node's index is i then:

$$\text{parent}(i) = \left\lfloor \frac{i}{2} \right\rfloor;$$ returns index of node's parent, e.g. parent(5)=2

$$\text{left}(i) = 2i;$$ returns index of node's left child, e.g. left(4)=8

$$\text{right}(i) = 2i + 1;$$ returns index of node's right child, e.g. right(4)=9



Operations with Heaps

- *Max_Heapify* (*A* , *i*)

Correct a single violation of the heap property occurring at the root *i* of an otherwise perfect subtree…

Setting: Assume that the trees rooted at left(*i*) and right(*i*) are max-heaps, but element A[*i*] violates the max-heap property;

i.e. A[*i*] is smaller than at least one of A[left(*i*)] or A[right(*i*)].

Goal: fix the subtree rooted at *i*.

Operations with Heaps

Max_heapify (A, *i*)

Find the index of the largest element among A[i], A[left(i)] *and* A[right(i)]

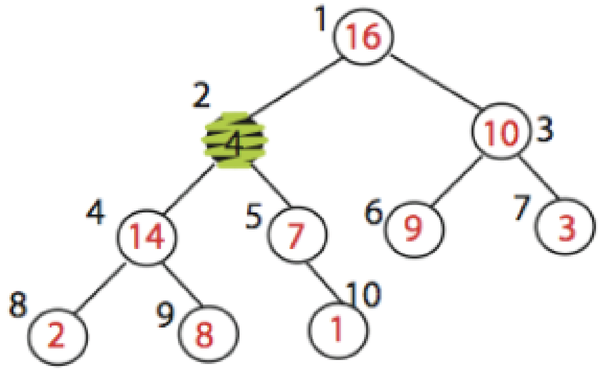
$l \leftarrow$ left(i)
$r \leftarrow$ right(i)
if $l \leq$ heap-size(A) and $A[l] > A[i]$
    then largest $\leftarrow l$
    else largest $\leftarrow i$
if $r \leq$ heap-size(A) and $A[r] > A[\text{largest}]$
    then largest $\leftarrow r$

If this index is different than i, exchange A[i] *with largest element; then recurse on subtree*

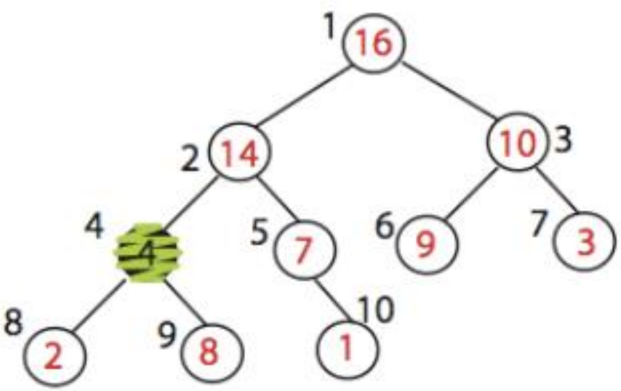
if largest $\neq i$
    then exchange $A[i]$ and $A[\text{largest}]$
        MAX_HEAPIFY(A, largest)

If A[i] is smaller than both A[left(i)] *and* A[right(i)] why do I insist on swapping with largest and not with any one of them, arbitrarily?

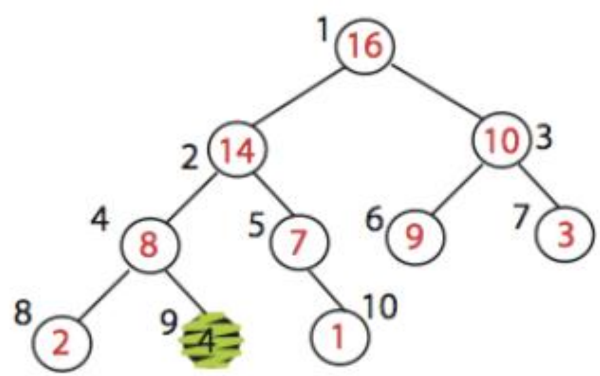
Max_Heapify (Example)



MAX_HEAPIFY (A,2)
heap_size[A] = 10



Exchange A[2] with A[4]
Call MAX_HEAPIFY(A,4)
because max_heap property
is violated



Exchange A[4] with A[9]
No more calls

Operations with Heaps

- *Max_Heapify* (A , i)

Correct a single violation of the heap property occurring at the root i of an otherwise perfect subtree.
Time O(log n).

- *Build_Max_Heap* (A)

Produce a max-heap from an unordered array A.

Operations with Heaps

Build_Max_Heap(A):

$\text{heap\_size}(A) = \text{length}(A)$

for $i \leftarrow \lfloor \text{length}[A]/2 \rfloor$ downto 1

do Max_Heapify(A, i)

Operation with Heaps

- *Max_Heapify* (*A* , *i*)

- Correct a single violation of the heap property occurring at the root *i* of an otherwise perfect subtree.
- Time O(log *n*).

- *Build_Max_Heap* (*A*)

- Produce a max-heap from an unordered array *A*.

- *Heapsort* (*A*)

- Sort an array *A* using heaps.

Operation with Heaps

HeapSort

1. Build Max Heap from unordered array;

2. Find maximum element A[1];

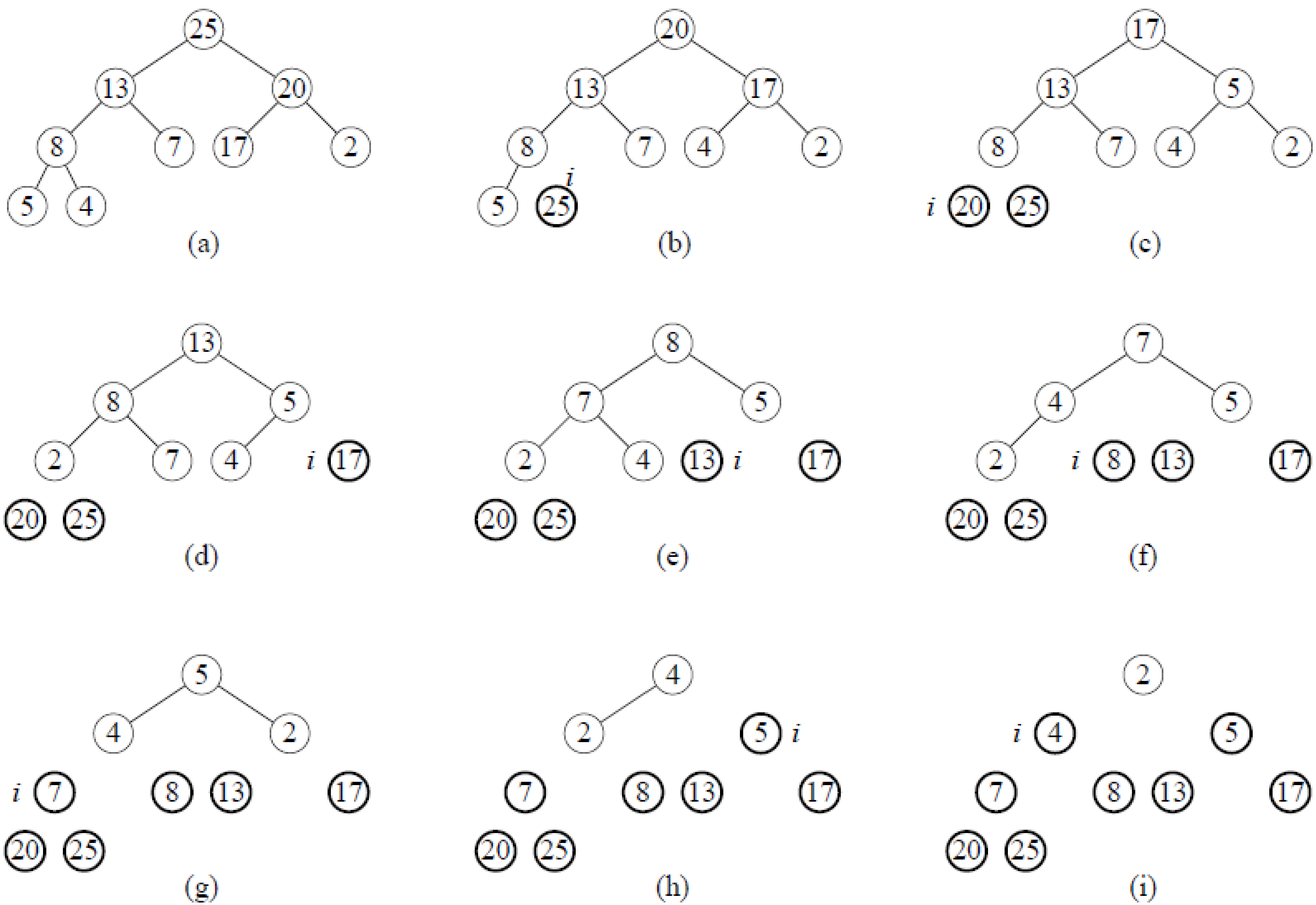
3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

4. Discard node n from heap
   (by decrementing heap-size variable)

5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.

6. Go to step 2.

Illustrate the operation of Heapsort on the array A[5,13,2,25,7,17,20,8,4]



Heap implementation of priority queue

- Heaps efficiently implement priority queues.
- Max-priority queues implemented with max-heaps. Min-priority queues are implemented with min-heaps similarly.

Priority queue

- is a data structure for maintaining a dynamic set S of elements, each with an associated value called a ***key***.

MAX-Priority Queue Operations

Max-priority queue supports the following operations:

- MAXIMUM(S) : returns element of S with largest key.
- EXTRACT-MAX(S): removes and returns element of S with largest key.
- INCREASE-KEY (S, x, k): increases value of element x's key to k. Assume k ≥x's current key value.
- INSERT(S , x): inserts element x into set S.

Finding the Max element

- Getting the maximum element is easy: it's the ROOT

```
HEAP-MAXIMUM(A)
  return A[1]
```

Extracting Max Element

Given the array A:

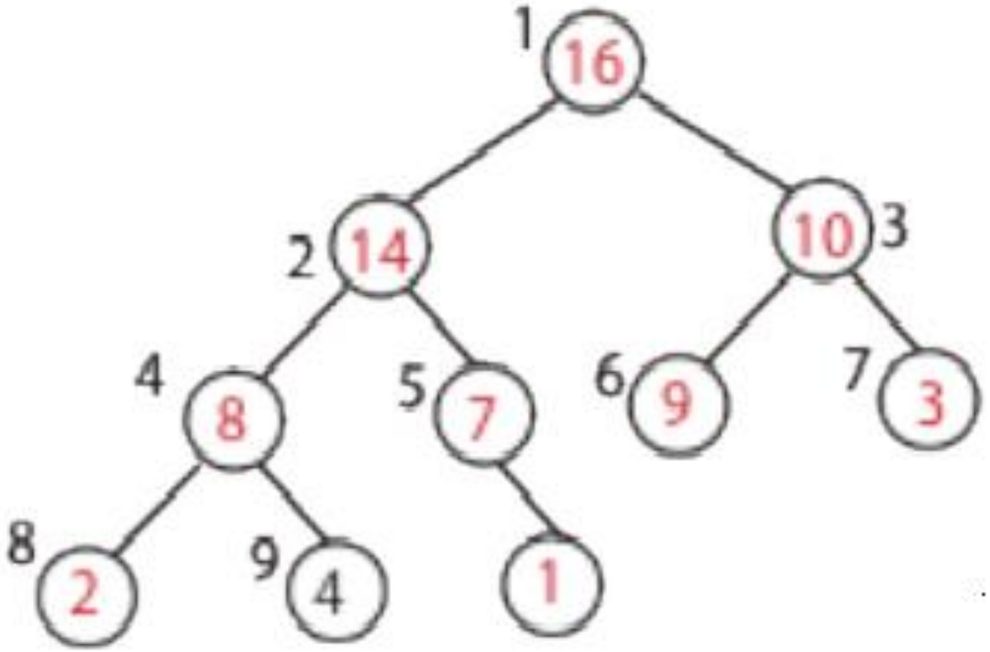
- Make sure heap is not empty.
- Make a copy of the maximum element (the root).
- Make the last node in the tree the new root.
- Re-heapify the heap, with one fewer node.
- Return the copy of the maximum element.

Extracting Max Element…

```
HEAP-EXTRACT-MAX(A, n)
  if n < 1
      error "heap underflow"
  max = A[1]
  A[1] = A[n]
  n = n - 1
  MAX-HEAPIFY(A, 1, n)          // remakes heap
  return max
```

EXAMPLE

Run HEAP-EXTRACT-MAX on the following heap



Increasing Key value

Given set S, element x, and new key value k:

- Make sure k ≥ x’s current key.
- Update x’s key value to k.
- Traverse the tree upward comparing x to its parent and swapping keys if necessary , until x’s key is smaller than its parent’s key.

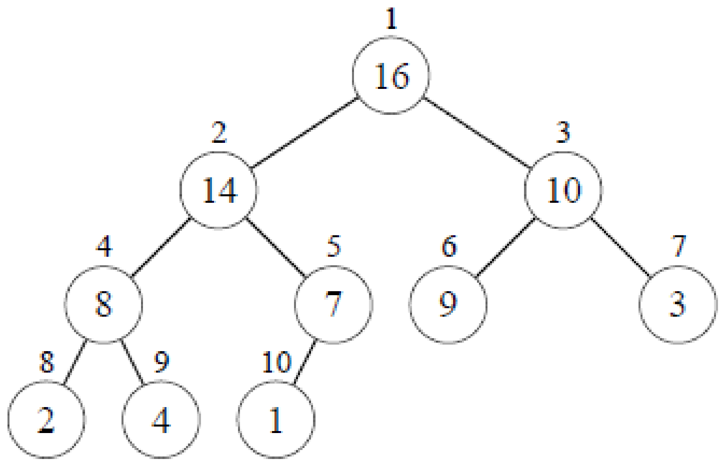
Increasing Key value…

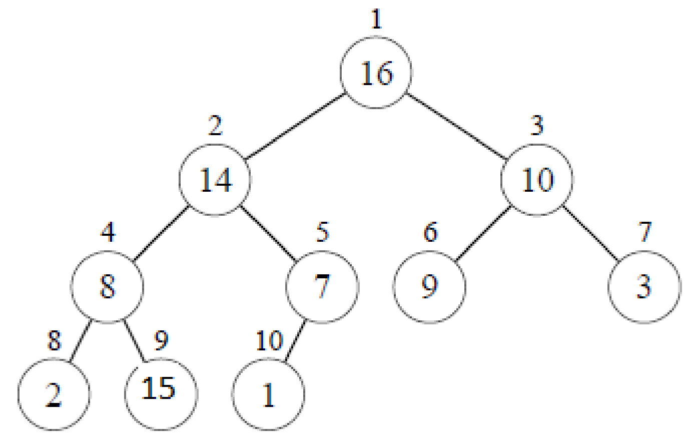
HEAP-INCREASE-KEY(A, i, key)

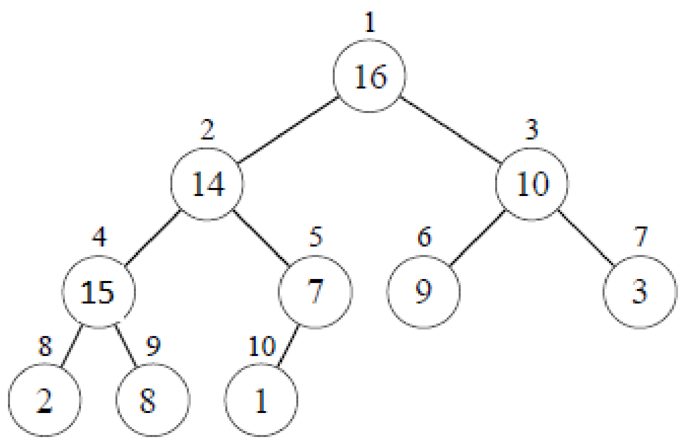
if $key < A[i]$
    error "new key is smaller than current key"
$A[i] = key$
while $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
    exchange $A[i]$ with $A[\text{PARENT}(i)]$
    $i = \text{PARENT}(i)$

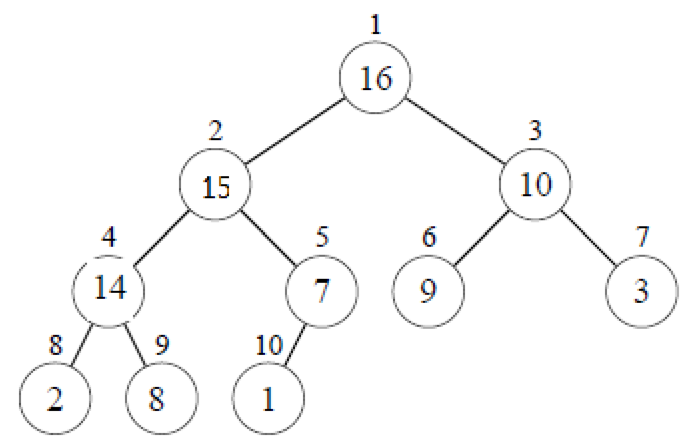
EXAMPLE

Increase key of node 9 in the following heap to have a value of 15.









Inserting into the heap

Given a key k to insert into the heap:

- Increment the heap size.
- Insert a new node in the last position in the heap, with key -∞.
- Increase the -∞ key to k using the HEAP-INCREASE-KEY procedure defined above.

Inserting into the heap…

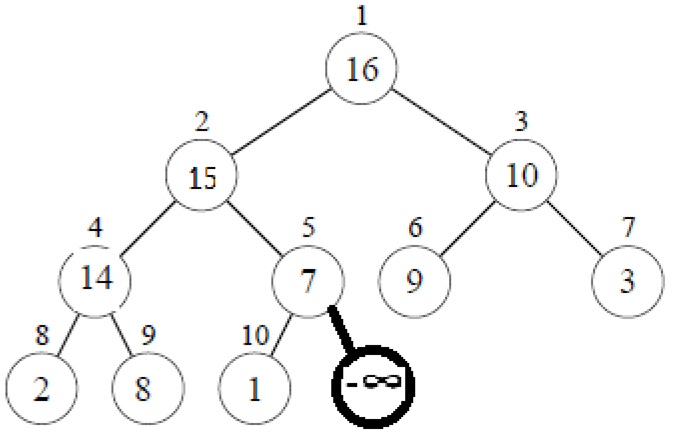
MAX-HEAP-INSERT(A, key, n)

$n = n + 1$

$A[n] = -\infty$

HEAP-INCREASE-KEY(A, n, key)

Example


1
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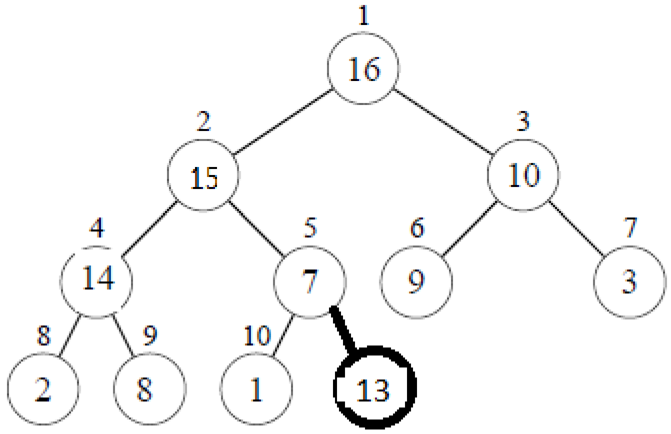
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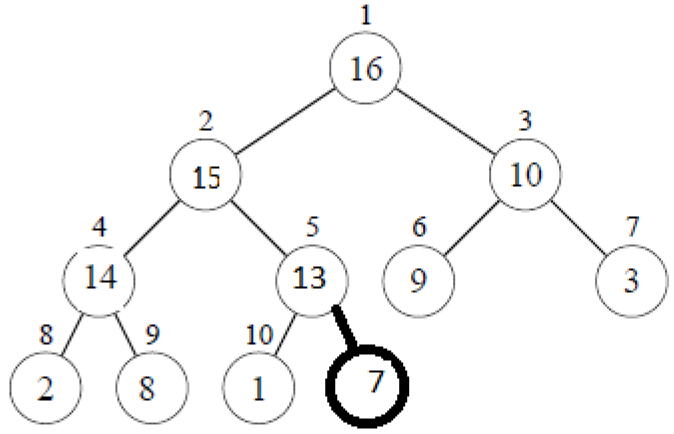
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Divide- and -conquer Next time !!!!